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Low momentum scattering of the Dirac particle with an asymmetric cusp potential

Yu Jiang[1,2,a], Shi-Hai Dong[2,b], A. Antillón[3], M. Lozada-Cassou[2]

[1] Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apartado Postal 55-334, 09340 D. F. México
[2] Programa de Ingeniería Molecular, Instituto Mexicano del Petróleo, Lázaro Cárdenas 152, 07730 México, D. F. México
[3] Centro de Ciencias Físicas, UNAM, Apartado Postal 48-3, 62251 Cuernavaca, Morelos, México

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Abstract. We study the exact solutions of the bound and scattering states of the one-dimensional Dirac equation with an asymmetric cusp potential and derive the condition of the supercriticality for this quantum system. We find that the scattering properties are invariant under reflection of the potential's shape, and the supercritical value for the potential amplitude V_0 varies with the degree of the potential asymmetry.

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1 Introduction

It is well known that the exact solutions of wave equations with certain physical potentials play an important role in quantum mechanics. For example, the exact solutions of the Schrödinger equation for a hydrogen atom and for a harmonic oscillator in three dimensions [1] were an important milestone at the beginning stage of quantum mechanics, which provided strong evidence supporting the correctness of the quantum theory. As we know, the study of the bound and scattering states of the quantum system has been known and understood well in the framework of the Schrödinger equation [2]. Generally speaking, for the low momentum scattering by the physical potentials, the transmission coefficient at zero energy becomes zero, but the reflection coefficient is unity unless the potential supports a half-bound state. The so-called half-bound state is described by the wave function that is finite at infinity but is not square integrable [2]. The transmission and reflection coefficients of the one-dimensional Schrödinger equation with a square potential barrier were first studied by Bohm [3], who showed that under certain conditions transmission resonance of the scattering states, characterized by transmission coefficient $T = 1$ and reflection coefficient $R = 0$, can be found for well-behaved potentials. After that, the transmission resonance has been carried out mainly in the framework of the one-dimensional nonrelativistic Schrödinger equation [4]. The characteristics of the transmission resonance in a quantum dot superlattice have also been studied [5].

Recently, the study of the bound and scattering states of the quantum system has been generalized to the relativistic equation case [6–10]. As we know, for some well-behaved potentials transmission resonance and supercritical states may be observed. It should be noted that most of the contributions mentioned above are concentrated on one-dimensional Dirac equation with symmetrical potentials such as the square well potential [6, 7], the Woods–Saxon one [8,9] and the cusp one [10]. Recently, studies on asymmetric potentials have been performed in [4,11]. For example, the bound and scattering states of the Dirac particle with an asymmetric potential formed by two unequal Dirac delta functions were studied in [11]. On the other hand, as addressed in the recent work on the low momentum scattering in the Dirac equation by Kennedy and Dombey [11], the transmission coefficient is always less than one, i.e., transmission resonance does not occur for asymmetric potentials. Actually, the study of the Dirac equation with the well-behaved potentials should go back to the investigation of Levinson's theorem in the 1980s [12], which is closely related to the half-bound state since Levinson's theorem establishes the relation between the number of bound states and the phase shifts at zero momentum. The study of the half-bound states has been discussed well in our previous works on Levinson's theorems for the Schrödinger equation [13], the Dirac equation [14] and the Klein–Gordon equation [15]. In particular, the strong Levinson theorem for the Dirac equation was established by Calogeracos and Dombey [14].

The purpose of this work is to obtain the exact solutions of the bound and scattering states for the one-dimensional Dirac equation with an asymmetrical cusp potential and to show that this potential with $V_0 < 0$ supports supercritical

[a] e-mail: jiang@xanum.uam.mx
[b] e-mail: dongsh2@yahoo.com

states. It is shown that as the momentum of the incident Dirac particle approaches zero, i.e., $E = m$ and $V_0 > 0$, the transmission coefficient T is not zero but satisfies $T < 1$, which coincides with the conclusion drawn by Kennedy and Dombey [11], which agrees also with the result for the Schrödinger equation [4]. It should be pointed out that the asymmetrical cusp potential is neither everywhere smooth nor of finite range.

This paper is organized as follows. Section 2 is devoted to studying the Dirac equation with the asymmetrical cusp potential. The exact solutions of the bound and scattering states are obtained. In Sect. 3 we derive the condition for the supercritical states. The concluding remarks are given in Sect. 4.

2 Dirac equation with asymmetrical cusp potential

We now study the Dirac equation with the asymmetrical cusp potential described by

$$U(x) = \begin{cases} V_0 \mathrm{e}^{x/a}, & \text{for} \quad x \leq 0, \\ V_0 \mathrm{e}^{-x/b}, & \text{for} \quad x \geq 0, \end{cases} \tag{1}$$

where V_0 shows the height of the barrier or the depth of the well. The parameters a and b describe the potential shape.

The Dirac equation in the presence of an external field A_μ can be written as [10]

$$[\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu) + m]\Psi = 0, \quad \mu = 0, 1. \tag{2}$$

We discuss the special case where only A_0 of A_μ is non-vanishing, i.e.,

$$eA_0 = U(x), \quad A_1 = 0. \tag{3}$$

By choosing the representation of the Dirac matrices $\gamma^0 = \mathrm{i}\sigma^2$ and $\gamma^1 = \sigma^1$, where σ^i $(i = 1, 2)$ are the Pauli matrices and taking

$$\Psi = \begin{pmatrix} F(x) \\ G(x) \end{pmatrix} \mathrm{e}^{-\mathrm{i}Et}$$

the Dirac equation $(\hbar = c = 1)$ becomes

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x} - \mathrm{i}[U(x) - E]F(x) + m\, G(x) = 0, \tag{4a}$$

$$\frac{\mathrm{d}G(x)}{\mathrm{d}x} + \mathrm{i}[U(x) - E]G(x) + m\, F(x) = 0, \tag{4b}$$

which can be solved exactly.

Introducing

$$\begin{aligned} y_a(x) &= 2\mathrm{i}eaV_0 \mathrm{e}^{x/a}, \quad \text{for} \quad x \leq 0, \\ y_b(x) &= 2\mathrm{i}ebV_0 \mathrm{e}^{-x/b}, \quad \text{for} \quad x \geq 0, \end{aligned} \tag{5}$$

we obtain the solutions in terms of the Whittaker function [16]

$$F(x) = \begin{cases} \dfrac{1/2 + \mu_b + k_b}{mb} N_2 y_b^{-1/2} M_{k_b+1,\mu_b}(y_b), \\ \quad \text{for } x > 0, \\ N_1 y_a^{-1/2} M_{k_a,\mu_a}(y_a), \\ \quad \text{for } x < 0, \end{cases} \tag{6a}$$

and

$$G(x) = \begin{cases} N_2 y_b^{-1/2} M_{k_b,\mu_b}(y_b), \\ \quad \text{for } x > 0, \\ -\dfrac{1/2 + \mu_a + k_a}{ma} N_1 y_a^{-1/2} M_{k_a+1,\mu_a}(y_a), \\ \quad \text{for } x < 0, \end{cases} \tag{6b}$$

with

$$\begin{aligned} k_a &= \mathrm{i}Ea - 1/2, \quad \mu_a = \mathrm{i}\sqrt{E^2 - m^2}\; a, \\ k_b &= \mathrm{i}Eb - 1/2, \quad \mu_b = \mathrm{i}\sqrt{E^2 - m^2}\; b. \end{aligned} \tag{7}$$

Here N_1 and N_2 are the non-trivial normalization constants. It should be noted that (6) will be used to derive the condition for the supercritical states below.

3 Supercritical states

We now discuss the bound and scattering states in order to derive the conditions for the supercritical states and study the relation between the reflection coefficient R and transmission coefficient T. By using the asymptotic form of the Whittaker function $M_{k,\mu}(z)$ [16]

$$M_{k,\mu}(z) \xrightarrow{z \to 0} \exp(-z/2) z^{1/2+\mu}, \tag{8}$$

it can be shown that the solutions of the Dirac equation (6) satisfy the boundary conditions that the spinor components $F(x)$ and $G(x)$ should vanish as $|x| \to \infty$. On the other hand, the energy levels may be obtained by imposing continuity of the spinor components $F(x)$ and $G(x)$ at $x = 0$. Applying this condition to the solutions (6), we find

$$\begin{aligned} &m^2 M_{k_a,\mu_a}(y_a) M_{k_b,\mu_b}(y_b) \\ &\quad -(\sqrt{E^2 - m^2} + E)^2 M_{k_a+1,\mu_a}(y_a) M_{k_b+1,\mu_b}(y_b) = 0, \end{aligned} \tag{9}$$

with

$$y_a = 2\mathrm{i}eaV_0, \quad y_b = 2\mathrm{i}ebV_0, \tag{10}$$

from which one can derive the condition for the supercritical states.

We first study the bound states $|E| < m$. We find the condition for supercritical states which implies that $E = -m$ and $|V_0| > 2m$ for purely attractive potentials $V_0 < 0$. But $V_0 > 2m$ does not necessarily mean that a particular half-bound state is supercritical. Even for $|V_0| > 2m$, half-bound states may exist where $E = m$. In this case we have $\mu_a = \mu_b = 0$ by setting $E = -m$. It is shown from (9) that

$$M_{k_a,0}(y_a)M_{k_b,0}(y_b) - M_{k_a+1,0}(y_a)M_{k_b+1,0}(y_b) = 0, \quad (11)$$

which implies that this relation remains invariant by interchanging the potential parameters. By making use of the condition of supercritical states (11) in a MATHEMATICA application, it is not difficult to calculate the supercritical values ($E = -m$, $V_0 < 0$ and $|V_0| > 2m$) and the critical values which refer to either $E = m$ or $E = -m$ and $|V_0| < 2m$[1]. We find for fixed a (or b) that $V_c(a,b)$, the critical value of $V_0(a,b)$, decreases with b (or a), and the supercritical value of the potential $V_{sc}(a,b)$ increases with b (or a), where the subscript sc refers to half-bound states of $E = -m$, not $E = m$. For example, at $(a,b) = (0.5, 0.5)$ we have $V_c \sim 1.88$, and $V_{sc} \sim -5.09$, while for $(a,b) = (0.5, 0.8)$ we have $V_c \sim 1.22$, and $V_{sc} \sim -4.45$, respectively. On the other hand, for $(a,b) = (0.5, 0.2)$, we have $V_c \sim 2.97$ and $V_{sc} \sim -6.51$, etc.

We now consider the scattering states. For an asymmetric potential, the scattering properties may differ for incoming waves from different directions. In view of the vanishing potential at infinity, $|x| \to \infty$, we assume that the incoming wave can be described by a plane wave. For the incident wave, from the large negative value of the x, we thus have

$$\Psi_{\text{inc.}} = \begin{pmatrix} y_a^{-1/2} M_{k_a,\mu_a}(y_a) \\ -\dfrac{1/2+\mu_a+k_a}{ma} y_a^{-1/2} M_{k_a+1,\mu_a}(y_a) \end{pmatrix} \stackrel{x\to-\infty}{\longrightarrow} \begin{pmatrix} 1 \\ -\dfrac{1/2+\mu_a+k_a}{ma} \end{pmatrix} \times (2\mathrm{i}eaV_0)^{\mu_a} \mathrm{e}^{\mathrm{i}x\sqrt{E^2-m^2}}. \quad (12)$$

Substituting μ_a by $-\mu_a$, we obtain the reflected wave

$$\Psi_{\text{refl.}} = \begin{pmatrix} y_a^{-1/2} M_{k_a,-\mu_a}(y_a) \\ -\dfrac{1/2-\mu_a+k_a}{ma} y_a^{-1/2} M_{k_a+1,-\mu_a}(y_a) \end{pmatrix} \stackrel{x\to-\infty}{\longrightarrow} \begin{pmatrix} 1 \\ -\dfrac{1/2-\mu_a+k_a}{ma} \end{pmatrix} \times (2\mathrm{i}eaV_0)^{-\mu_a} \mathrm{e}^{-\mathrm{i}x\sqrt{E^2-m^2}}. \quad (13)$$

On the other hand, the transmitted wave has the form

$$\Psi_{\text{trans.}} = \begin{pmatrix} \dfrac{1/2-\mu_b+k_b}{mb} y_b^{-1/2} M_{k_b+1,-\mu_b}(y_b) \\ y_b^{-1/2} M_{k_b,-\mu_b}(y_b) \end{pmatrix} \stackrel{x\to\infty}{\longrightarrow} \begin{pmatrix} \dfrac{1/2-\mu_b+k_b}{mb} \\ 1 \end{pmatrix} \times (2\mathrm{i}ebV_0)^{-\mu_b} \mathrm{e}^{\mathrm{i}x\sqrt{E^2-m^2}}. \quad (14)$$

The reflection and transmission coefficients can be obtained by continuity of the spinor function Ψ at $x = 0$:

$$A\Psi_{\text{inc.}}(x=0) + B\Psi_{\text{refl.}}(x=0) = C\Psi_{\text{trans.}}(x=0). \quad (15)$$

The ratio between the reflection and the incidence coefficients is given by

$$\frac{B}{A} = \left(M_{k_a,\mu_a}(y_a)M_{k_b,-\mu_b}(y_b) - M_{k_a+1,\mu_a}(y_a)M_{k_b+1,-\mu_b}(y_b)\right) \Big/ \left(-\frac{1/2-\mu_a+k_a}{ma}\frac{1/2-\mu_b+k_b}{mb} \times M_{k_a+1,-\mu_a}(y_a)M_{k_b+1,-\mu_b}(y_b) - M_{k_a,-\mu_a}(y_a)M_{k_b,-\mu_b}(y_b) \right), \quad (16)$$

where we have used (7).

Since both A and B tend to a vanishing value as μ_a and μ_b approach zero, the limit for $E = \pm m$ should be evaluated from their derivatives. We use MATHEMATICA to calculate the limiting behavior of A and B and find that for asymmetric potential, the transmission coefficient T is always less than one at $E = m$ for an asymmetric potential which supports a half-bound state, which coincides with the result of Kennedy and Dombey [11]. In Fig. 1 we plot

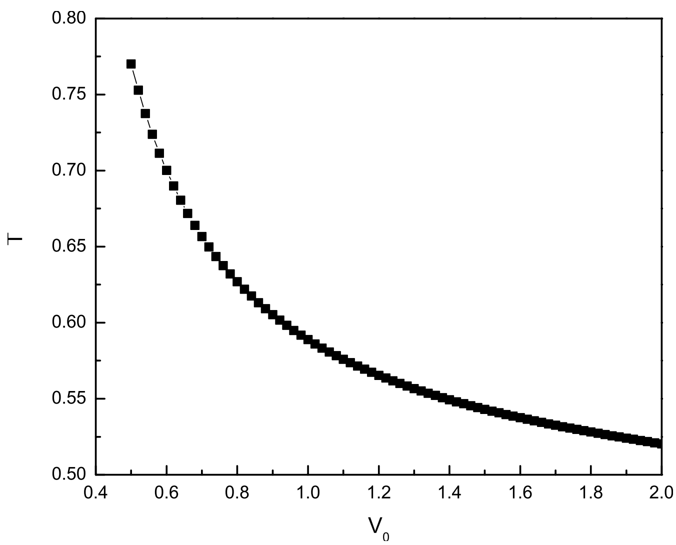

Fig. 1. The transmission coefficient T as a function of the potential strength V_0. Parameters are taken as $a = 0.5$, $b = 1.0$, $m = E = 0.8$

[1] We may plot the real and imaginary parts of the condition (11) as a function of V_0. The supercritical value corresponds to the first zero of (11) with $V_0 < 0$, while the critical value of the potential is the first zero of (11) with $|V_0| < 2m$.

the transmission coefficient T as a function of the potential strength for the asymmetric potential constants $a = 0.5$, $b = 1.0$, $m = E = 0.8$.

4 Concluding remarks

In this work we have obtained the exact solutions of the bound and scattering states for the one-dimensional Dirac equation with the asymmetrical cusp potential and derived the condition for the supercritical states. We find that supercritical states ($V_0 < 0$ and $|V_0| > 2m$) and a complete reflection of this quantum system occur at $E = -m$ and at $E = \pm m$, respectively. On the other hand, we have found that the condition for the supercritical states keeps symmetry by interchanging the potential parameters a and b. We have also shown that at zero momentum, the transmission coefficient $T = 0$ identically for the cusp potentials, which in turn reveals some peculiar characteristics of this discontinuous and long-range potential.

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